Lecture 8: Entanglement

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Lecturer: Saeed Mehraban *Scribe: Preliminary notes*

1 Non-classical correlations

If a quantum state is the tensor product of two quantum states, we call the quantum state separable. In general, quantum states are not separable. We call such a quantum state "entangled". For instance, let $|\psi_1\rangle = \alpha|0\rangle + \beta|1\rangle$ and $|\psi_2\rangle = \gamma|0\rangle + \delta|1\rangle$. The quantum state $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$ is a separable quantum state. If we expand $|\psi\rangle = \alpha\gamma|00\rangle + \alpha\delta|01\rangle + \beta\gamma|10\rangle + \beta\delta|11\rangle$. In general, Let α_x be the amplitude corresponding to $|x\rangle$. We see that if these amplitudes satisfy $\alpha_{00}\alpha_{11} = \alpha_{01}\alpha_{10}$, then the quantum state is entangled. There is a sense to say a quantum state is maximally entangled. Consider the example of EPR pairs below.

- EPR pair $|\Phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

Exercise: Show that the EPR pair is not separable.
EPR pairs satisfy the following surprising relationsships

- EPR pair $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(|++\rangle + |--\rangle)$
- $|00\rangle + |11\rangle = |\theta\uparrow, \theta\uparrow\rangle + |\theta\downarrow, \theta\downarrow\rangle$

Here

- $|\theta\uparrow\rangle = \cos\theta|0\rangle + \sin\theta|1\rangle$
- $|\theta\downarrow\rangle = -\sin\theta|0\rangle + \cos\theta|1\rangle$

In general for any square matrix M we have $(M \otimes I)|\Phi\rangle = (I \otimes M^T)|\Phi\rangle$.

2 Quantum teleportation

One of the main implications of entanglement is quantum teleportation: We can destroy a quantum state at one point in space and recreate it somewhere else if we are allowed to send a few classical bits. Here is how it is done. First, observe that the following states are complete basis for two quantum bits.

$$|\Phi_\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle), \quad |\Psi_\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |01\rangle).$$

We can show that

$$|\Phi_+\rangle\langle\Phi_+| + |\Phi_-\rangle\langle\Phi_-| + |\Psi_+\rangle\langle\Psi_+| + |\Psi_-\rangle\langle\Psi_-| = I$$

So $Q = \{|\Phi_+\rangle\langle\Phi_+|, |\Phi_-\rangle\langle\Phi_-|, |\Psi_+\rangle\langle\Psi_+|, |\Psi_-\rangle\langle\Psi_-|\}$ forms a POVM. It is useful to note

$$|00\rangle = \frac{1}{\sqrt{2}}(|\Phi_+\rangle + |\Phi_-\rangle), \quad |11\rangle = \frac{1}{\sqrt{2}}(|\Phi_+\rangle - |\Phi_-\rangle),$$

$$|01\rangle = \frac{1}{\sqrt{2}}(|\Psi_+\rangle + |\Psi_-\rangle), \quad |10\rangle = \frac{1}{\sqrt{2}}(|\Psi_+\rangle - |\Psi_-\rangle).$$

The teleportation protocol is as follows: Alice wishes to send a quantum bit $|\psi\rangle$ to Bob; A be a register at Alice's side and B be a register at Bob's, where he wishes to receive the quantum state. She stores $|\psi\rangle$ in a separate register C and shares an EPR state ($|\Phi_+\rangle_{AB}$) with Bob. Alice measures (destroys) the registers AC in according to POVM Q. Let $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$. With probability $1/4$ the content of ABC will be either of the following:

$$|\Phi_+\rangle_{AC} \otimes (\alpha|0\rangle + \beta|1\rangle)_B, \quad |\Phi_-\rangle_{AC} \otimes (\alpha|0\rangle - \beta|1\rangle)_B,$$

$$|\Psi_+\rangle_{AC} \otimes (\alpha|1\rangle + \beta|0\rangle)_B, \quad |\Psi_-\rangle_{AC} \otimes (\alpha|1\rangle - \beta|0\rangle)_B,$$

We observe that in either of these cases Bob has received $|\psi\rangle$ up to some error. The last step is for Bob to correct this error. Here is how he can do it. Alice selects two classical bits b_X, b_Z. Upon measuring $|\Phi_+\rangle_{AC}$ she sets $b_X = 0, b_Z = 0$; upon measuring $|\Phi_-\rangle_{AC}$ she sets $b_X = 0, b_Z = 1$; upon measuring $|\Psi_+\rangle_{AC}$ she sets $b_X = 1, b_Z = 0$; upon measuring $|\Psi_-\rangle_{AC}$ she sets $b_X = 1, b_Z = 1$. She sends b_X, b_Z to Bob. Bob corrects the error by applying $Z^{b_Z}X^{b_X}$. In conclusion, by sending two classical bits, and sharing an entangled state, Alice can send 1 quantum bit to Bob.

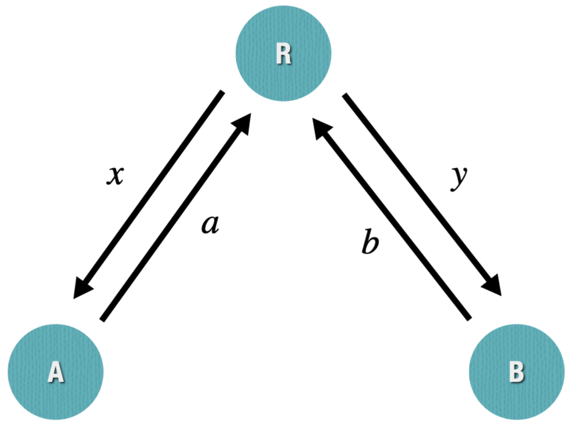
3 CHSH game

We saw that quantum superposition has a probabilistic behavior. For instance, if we measure the $|+\rangle$ in the $|0\rangle, |1\rangle$ basis, we obtain 0 or 1, each with probability $1/2$. What is the value of the quantum state before we measure this quantum state? Local realism is the idea that physical systems have definite properties independent of measurement and that these properties can influence the outcomes of measurements in a local manner. All classical computations are based on the local realism framework. The CHSH inequality, named after its discoverers Clauser, Horne, Shimony, and Holt, is a fundamental result in quantum mechanics that demonstrates a violation of local realism.

In this lecture, we discuss a certain quantum mechanical "game" that is inspired by the CHSH inequality. We show how the predictions of this game violate local realism. This game has been performed experimentally, confirming the violation of local realism. The outcome of this experiment indicates that the nature of correlations in quantum mechanics is fundamentally different from our classical intuition. We note that while this experiment violates "local" realism, there might be "non-local" hidden variables that justify the probabilistic nature of quantum mechanics. However, the experiment does refute local hidden variables.

Game:

Figure 1: The description of the CHSH game



- R gives A and B two bits x, y
- they don't interact
- they output a and b back to R.
- they win if x AND $y = a \oplus b$.

Exercise: Show that classical algorithms can win with at most 75%.
Exercise: Show that if one player measures the EPR pair in θ basis and another in ϕ basis, then they output the same value w.p $\cos^2(\theta - \phi)$.

Turns out there is a quantum algorithm that wins with probability $\cos^2 \pi/8 \approx 85\%$. Here is how

- A:
  - If receives $x = 0$ they measure in $\theta = 0$ basis
  - If receives $x = 1$ they measure in $\theta = \pi/4$ basis
- B
  - If receives $y = 0$ they measure in $\theta = \pi/8$ basis
  - If receives $y = 1$ they measure in $\theta = -\pi/8$ basis

Can show they win with the mentioned probability. Implications to local hidden variable theories.

Figure 2: Alice and Bob strategies

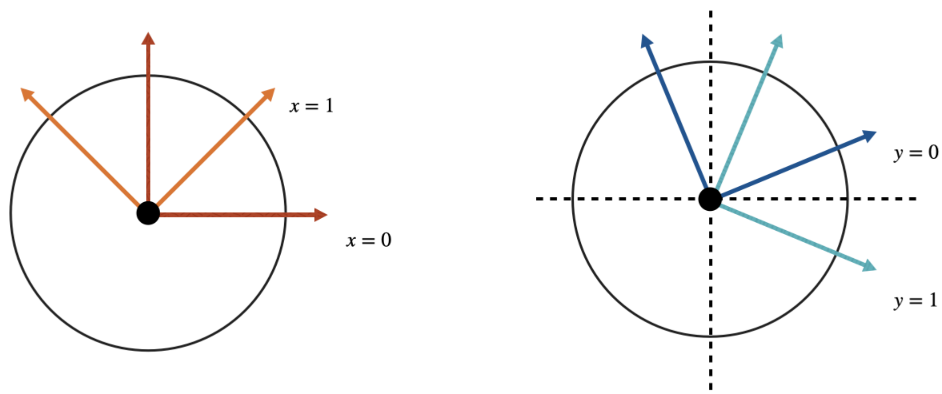
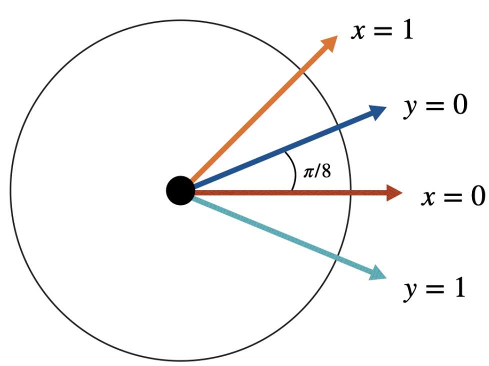


Figure 3: Analysis of the strategies



- $\angle(x = 0, y = 0) = \pi/8$. Output $a = b$ w.p. $\cos^2 \pi/8$
- $\angle(x = 0, y = 1) = \pi/8$. Output $a = b$ w.p. $\cos^2 \pi/8$
- $\angle(x = 1, y = 0) = \pi/8$. Output $a = b$ w.p. $\cos^2 \pi/8$
- $\angle(x = 1, y = 1) = \pi/4 + \pi/8 = \pi/2 - \pi/8$. Output $a \neq b$ wp $\cos^2 \pi/8$

Exercise: Can you win with higher probability by any quantum strategy?